Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods I

Thursday, 16th January 2014, 9:30 a.m.- 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Section A — Mathematical Methods

Question 1

(a) (i) Find the general solution of the Cauchy Euler differential equation:

$$Ly \equiv x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0, \tag{1}$$

for $1 < x < 2$.

[6 marks]

(ii) Consider the boundary value problem

$$Ly(x) = f(x), \quad 1 < x < 2, \qquad y(1) - \frac{\mathrm{d}y}{\mathrm{d}x}(1) = 0, \quad y(2) = 0, \tag{2}$$

with Ly as in (1). Write down two equivalent problems for the Green's function $g(x, \xi)$:

(I) using the delta function $\delta(x)$;

(II) using only classical functions and with appropriate conditions at $x = \xi$.

You do not need to compute $g(x, \xi)$ explicitly. **[6 marks]**

(b) The set C_0^∞ denotes the set of "test" functions, i.e. all functions that have compact support and have derivatives of arbitrary order.

(i) Define what it means to say that the map $T : C_0^\infty \to \mathbb{R}$ is a *distribution*. **[5 marks]**

(ii) Show that $x\delta'(x) = -\delta(x)$ *in the distributional sense.* **[8 marks]**

Question 2

The integral operator L is defined by

$$Ly \equiv \int_0^\pi k(x,t)y(t)\ \mathrm{d}t$$

where $k(x,t)$ is real and continuous.

(a) If $k(x,t) = -k(t,x)$, show that $L^* = -L$ and that the eigenvalues λ_k satisfying

$$Ly_k = \lambda_k y_k$$

are purely imaginary.

[8 marks]

(b) Find the eigenvalues of finite multiplicity, and the corresponding eigenfunctions, when $k(x,t) = \sin 2x + \sin 2t$. Give an example of an eigenfunction corresponding to $\lambda = 0$.

[9 marks]

(c) Consider the integral equation $Ly = \alpha y + \cos x$. Determine for which real α there is a unique solution $y(x)$ (you do not need to find it).

[8 marks]

Question 3

The differential operator L is defined by

$$Ly \equiv y''(x) - 2y'(x) + y(x) \tag{3}$$

(a) Show that

$$\lambda_k = \pi^2 k^2, \quad y_k = e^x \cos(k\pi x), \quad k = 0, 1, 2, \ldots$$

are eigenvalues and eigenfunctions of $Ly_k = -\lambda_k y_k$, given the boundary conditions

$$y(0) - y'(0) = 0, \quad y(1) - y'(1) = 0. \tag{4}$$

[8 marks]

(b) Find the adjoint problem for the operator (3) and boundary conditions (4).

[7 marks]

(c) You may use without proof that $w_k = e^{-x} \cos(k\pi x)$ are adjoint eigenfunctions. Use the eigenfunctions and their adjoints to obtain a formula for the coefficients in an eigenfunction expansion

$$y = \sum_{n=0}^{\infty} c_k y_k(x)$$

for the solution of the problem

$$y''(x) - 2y'(x) + ay(x) = e^{2x},$$
$$y(0) - y'(0) = 3, \quad y(1) - y'(1) = 2,$$

where $a < 0$ is a real constant.
You may assume, without proof, that all eigenvalues/eigenfunctions are as listed in part (a) above. Also, you do not need to evaluate the integrals in the formulae for c_k.

[10 marks]

Question 4

We consider here polynomial solutions of *Hermite's equation*

$$y'' - 2xy' + 2ny = 0, \tag{5}$$

where n is a non-negative integer.

(a) Classify $x = \infty$ as an ordinary, regular singular, or irregular singular point of the ODE, giving reasons for your answer.

[10 marks]

(b) The Rodrigues' formula for the nth order Hermite polynomials is given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2} \right). \tag{6}$$

Prove that H_n given by (6) is a solution of Hermite's equation (5).

[7 marks]

(c) By converting to Sturm-Liouville form, obtain an orthogonality relation for the Hermite polynomials. Using (without proof) that the region of integration is $(-\infty, \infty)$, verify the orthogonality relation for functions $H_n(x)$, $H_m(x)$ given by (6) when $n \neq m$.

[8 marks]

Section B — Further Mathematical Methods

Question 5

Suppose the functions x and y minimise the functional

$$J[x,y] = \int_0^T F(t,x,\dot{x},y,\dot{y})\,\mathrm{d}t,$$

over all $x, y \in C^2[0,T]$, subject to $x(0) = a$, $y(0) = b$, $x(T) = c$, $y(T) = d$, where F is continuously differentiable, and a dot represents $\mathrm{d}/\mathrm{d}t$.

(a) Show that x and y satisfy the Euler equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{x}}\right) - \frac{\partial F}{\partial x} &= 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{y}}\right) - \frac{\partial F}{\partial y} &= 0. \end{aligned}$$

[9 marks]

(b) The *Hamiltonian*, H, is given by

$$H = \dot{x}\frac{\partial F}{\partial \dot{x}} + \dot{y}\frac{\partial F}{\partial \dot{y}} - F.$$

Show that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial F}{\partial t}.$$

[4 marks]

(c) Two unit masses at positions $x(t)$ and $y(t)$ are connected by a spring with spring constant k and natural length 1. They are to be moved from $x(0) = 0$, $y(0) = 1$ to $x(T) = 1$, $y(T) = 4$ in such a way as to minimise the total energy

$$\int_0^T \left[\dot{x}^2 + \dot{y}^2 + k(y - x - 1)^2\right]\mathrm{d}t.$$

(i) Show that the average velocity $(\dot{x} + \dot{y})/2$ is constant during the motion.

(ii) Find the optimal trajectories $x(t)$ and $y(t)$.

[12 marks]

TURN OVER

Question 6

(a) Find the stationary points of the system

$$\begin{aligned}\dot{x} &= \mu y + y^3,\\ \dot{y} &= -x + (\mu - 1)y - y^3.\end{aligned}$$

Classify the stationary point at the origin as a function of the parameter μ.

What kind of bifurcation happens at $\mu = 0$?

What kind of bifurcation happens at $\mu = 1$?

[12 marks]

(b) Now suppose $\mu = 1 + \epsilon$. By writing $x = \epsilon^{1/2}X$, $y = \epsilon^{1/2}Y$ use the Poincaré-Lindstedt method to find the periodic orbit when $0 < \epsilon \ll 1$. Show that the frequency of the oscillation is

$$\omega = 1 + \epsilon + O(\epsilon^2),$$

and find the leading-order amplitude.

[13 marks]

[*You may use the identities*

$$\sin^3\tau = \frac{3\sin\tau}{4} - \frac{\sin 3\tau}{4}, \qquad \sin^2\tau\cos\tau = \frac{\cos\tau}{4} - \frac{\cos 3\tau}{4},$$

$$\cos^2\tau\sin\tau = \frac{\sin\tau}{4} + \frac{\sin 3\tau}{4}, \qquad \cos^3\tau = \frac{3\cos\tau}{4} + \frac{\cos 3\tau}{4},$$

without proof.]